In the Name of GOD

Our classes will be held in person From this week. Please do what you must do to Come to Iran and join the class in person.

Band-pass stochastic Processes

- How to represent based on equivalent low-pass Process.

As you now, the power spectral density (PSD) of any stochastic process in

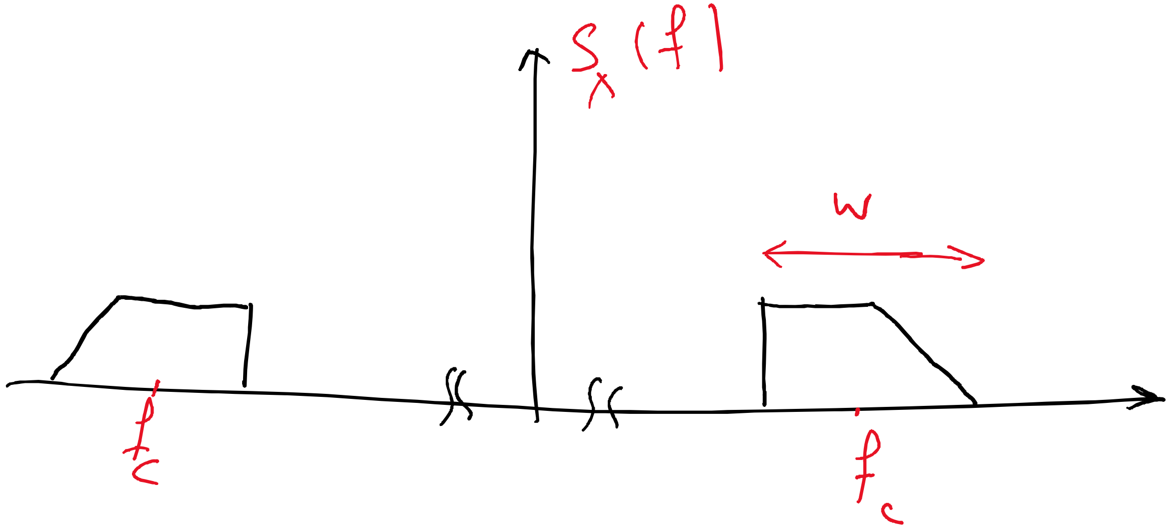
$$FT\left\{ \left\langle \underbrace{R_x(t+\tau, t)} \right\rangle_t \right\} = S_x(f)$$

In this course we work with the WSS stochastic Processes with zero mean.

WSS $\Longrightarrow$ stochastic Process $\quad R_x(t+\tau, t) = R_x(\tau)$

$$\Longrightarrow \quad S_x(f) = FT\{R_x(\tau)\}, \quad m_x(t) = 0$$

A bandpass stochastic is that, its PSD is around a central frenency (f_c), that is very larger than stachastic process bandwidth.



$f_c \gg W$

narrow band

In this course we consider a bandpass (narrow band) stochastic process $n(t)$ with zero mean and power spectral density $\Phi_n(f)$

we want to find a representation of $n(t)$ based on its equivalent low-pass sto. process. So, we define these relations

$$\underbrace{n(t)}_{\text{band-pass stochastic process}} = \mathrm{Re}\left\{ \underbrace{z(t)}_{\text{equivalent low-pass stochastic process}} e^{j2\pi f_c t} \right\} \rightarrow \text{rep} \quad (1)$$

$$n(t) = \underbrace{\overbrace{x(t)}^{\text{stochastic process}}}_{\text{Inphase part of } n(t)} \cos 2\pi f_c t - \underbrace{\overbrace{y(t)}^{\text{stochastic process}}}_{\text{Quadrature part of } n(t)} \sin 2\pi f_c t \longrightarrow \text{rep } (2)$$

in which $z(t) = x(t) + j\,y(t)$

$$n(t) = \underbrace{a(t)} \cos\left(2\pi f_c t + \underbrace{\theta(t)}\right)$$

rep (3)

stochastic processes

in which $a(t) = \sqrt{x^2(t) + y^2(t)}$, $\theta(t) = \tan^{-1} \dfrac{y(t)}{x(t)}$

from rep 2, we can conclude that $x(t)$ and $y(t)$

are also zero mean stochastic processes

$$\overbrace{E\{n(t)\}}^{0} = E\{x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t\}$$

$$= E\{x(t)\}\cos 2\pi f_c t - E\{y(t)\}\sin 2\pi f_c t \equiv 0$$

$$\Longrightarrow \quad E\{x(t)\} = 0 \quad \text{and} \quad E\{y(t)\} = 0$$

This is our first conclusion about $x(t)$ and $y(t)$ (Quadrature parts of $n(t)$) that is the first property of $x(t)$ and $y(t)$.

As you know the PSD of any WSS stochastic processes is the Fourier Transform of its correlation function. So, the first step for finding the representation of $n(t)$ based on its equ. low-pass sto. process, is to find a relation for their

Correlation functions.

As you know, we have

$$R_n(\tau) = E\{n(t+\tau)\, n^*(t)\}$$

(n(t) is a bandpass stochastic process and it is real)

$$R_n(\tau) = E\{\underbrace{n(t+\tau)}\,\underbrace{n(t)}\}$$

$$= E\Big\{\overbrace{\big[x(t+\tau)\cos 2\pi f_c(t+\tau) - y(t+\tau)\sin 2\pi f_c(t+\tau)\big]}^{n(t+\tau)}$$

$$\times \underbrace{\big[x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t\big]}_{n(t)}\Big\}$$

rep(2)

trigonometric $\Longrightarrow$ equalities to simplify the relation $(*)$ (do as an assignment)

$$\underline{R_n(\tau)} = E\{x(t+\tau)\,x(t)\}\underbrace{\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t}_{(*)}$$

(Does not depend on t)

$$+\,E\{y(t+\tau)\,y(t)\}\underbrace{\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t}_{(*)}$$

$$-\,E\{y(t+\tau)\,x(t)\}\underbrace{\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t}_{(*)}$$

$$-\,E\{x(t+\tau)\,y(t)\}\underbrace{\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t}_{(*)}$$

The main Concept that help us to simplify this relation is that the right part of the relation does not depend on t. So the left part of this relation must not depend on t. So we can write

$$E\{x(t+\tau)\,x(t)\} = \varphi_x(\tau)$$ (does not depend on t)

$$E\{y(t+\tau)\,y(t)\} = \varphi_y(\tau)$$ (")

$$E\{x(t+\tau)\,y(t)\} = \varphi_{xy}(\tau)$$ (")

$$E\{y(t+\tau)\,x(t)\} = \varphi_{yx}(\tau)$$ (")

These relations show us that $x(t)$ and $y(t)$ are jointly WSS stochastic processes. This is the second property of $x(t)$ and $y(t)$.

$x(t)$ and $y(t)$ are also zero mean

$\Longrightarrow$ Quadrature parts of $n(t)$ are zero-mean jointly WSS stochastic processes.

$x(t)$ and $y(t)$ are jointly WSS sto. process

$\Longrightarrow$ $x(t)$ is WSS stoch. process and $y(t)$ is WSS st. process.

$$\Longrightarrow \varphi_n(\tau) = \varphi_x(\tau)\underbrace{\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t}_{(*)}$$

$$+\ \varphi_y(\tau)\underbrace{\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t}_{(*)}$$

$$-\ \varphi_{yx}(\tau)\underbrace{\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t}_{(*)}$$

$$-\ \varphi_{xy}(\tau)\underbrace{\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t}_{(*)}$$

use trigonometric equalities to simplify these relations

$$\Longrightarrow \varphi_n(\tau) = \frac{1}{2}\left[\varphi_x(\tau) + \varphi_y(\tau)\right]\cos 2\pi f_c \tau$$

Does not depend on t

$$- \frac{1}{2}\underbrace{\left[\varphi_x(\tau) - \varphi_y(\tau)\right]}\cos(4\pi f_c t + \tau)$$

$$- \frac{1}{2}\left[\varphi_{yx}(\tau) - \varphi_{xy}(\tau)\right]\sin 2\pi f_c \tau$$

$$- \frac{1}{2}\underbrace{\left[\varphi_{yx}(\tau) + \varphi_{xy}(\tau)\right]}\sin(4\pi f_c t + \tau)$$

$$\Longrightarrow \quad \varphi_x(\tau) = \varphi_y(\tau) \quad \text{and} \quad \varphi_{yx}(\tau) = -\varphi_{xy}(\tau)$$

These relations show us the third property of $x(t)$ and $y(t)$. Means that the quadrature part of $n(t)$ have similar correlation function

$\varphi_x(\tau) = \varphi_y(\tau)$. So, the PSD of $x(t)$ and $y(t)$ are also similar.

$$FT\{\varphi_x(\tau)\} = FT\{\varphi_y(\tau)\} \Rightarrow \Phi_x(f) = \Phi_y(f)$$

and $$\varphi_{xy}(\tau) = -\varphi_{yx}(\tau)$$

$$\Rightarrow \quad \varphi_n(\tau) = \varphi_x(\tau)\cos 2\pi f_c \tau - \varphi_{yx}(\tau)\sin 2\pi f_c \tau$$

As you see, this relation is similar to the relation we have for Deter. Signals (I) but based on Correlation functions of $x(t)$ and $y(t)$. So this relation can help us to find the rep. of $n(t)$ based on eqv. low-pass stoch. process.

$$s(t) = x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t \quad (I)$$

Means that the similar relation between det. signal and its quadrature parts are also is true for Correlation function of an stoch. Process and that of its quadrature

Correlation functions

parts.

This is our key to find a rep. for band-pass stoch. process $n(t)$ based on its eqv. low-pass stoch. Process.